A COMMENT ON FREE-FERMION CONDITIONS
FOR LATTICE MODELS IN TWO AND MORE DIMENSIONS

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Abstract. We analyze free-fermion conditions on vertex models. We show –by examining examples of vertex models on square, triangular, and cubic lattices– how they amount to degeneration conditions for known symmetries of the Boltzmann weights, and propose a general scheme for such a process in two and more dimensions.

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1 Introduction

We consider vertex models in D dimensions. As is standard, the bonds of the lattice carry variables taking q values (colors). The model is determined by attributing Boltzmann weights to the various possible bond configurations around a vertex [1]. These homogeneous weights are arranged in a matrix, which we denote by R. The size and form of the matrix R vary according to the number of colors, and the coordination number of the lattice. Typical examples we will consider are the 2D square lattice, the 2D triangular lattice, and the 3D cubic lattice:

| 2D square | 2D triangular | 3D cubic |
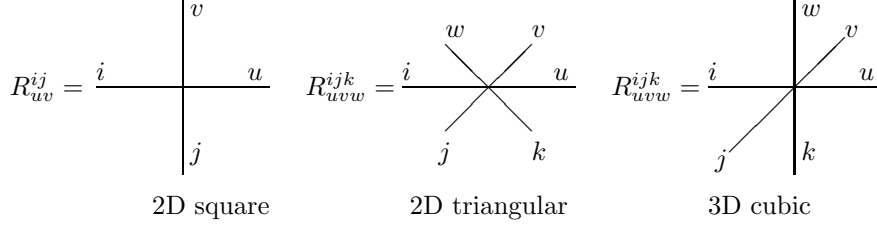
If the number of colors q is 2, and we will restrict ourselves to this case, then the R-matrices are of sizes 4×4, 8×8, and 8×8 respectively. The difference between 2D triangular and 3D cubic for example does not show in the size nor the form of the matrix. It will, however, appear in the operations we define on the matrices.

We shall use a number of elementary transformations acting on the matrices. These transformations come from the inversion relations and the geometrical symmetries of the lattice, in the framework of integrability [2, 3, 4, 5], and beyond integrability [6]. They generically form an infinite group $\Gamma_{lattice}$ [4, 5].

The groups $\Gamma_{lattice}$ have a finite number of involutive generators. The first one, denoted I, is non-linear and does not depend on the lattice: it is the matrix inversion up to a factor. The other generators act linearly on R, actually by permutations of the entries, and represent the geometrical symmetries of the lattice.

For the square lattice, we have two linear transformations, the partial transpositions t_l and t_r [5]:

$$(t_l R)^{ij}_{uv} = R^{uj}_{iv}, \qquad (t_r R)^{ij}_{uv} = R^{iv}_{uj} \qquad i,j,u,v = 1..q \qquad (1)$$

The product $t_l t_r$ is the matrix transposition (l stand for 'left' and r stands for 'right' in the standard tensor product structure of R).

For the triangular lattice, we have three linear transformations τ_l, τ_m, τ_r:

$$(\tau_l R)^{ijk}_{uvw} = R^{ukj}_{iwv}, \qquad (\tau_m R)^{ijk}_{uvw} = R^{wvu}_{kji}, \qquad (\tau_r R)^{ijk}_{uvw} = R^{jiw}_{vuk}. \qquad (2)$$

Finally, for the cubic lattice, we have three linear transformations t_l, t_m, t_r [5]:

$$(t_l R)^{ijk}_{uvw} = R^{ujk}_{ivw}, \quad (t_m R)^{ijk}_{uvw} = R^{ivk}_{ujw}, \quad (t_r R)^{ijk}_{uvw} = R^{ijw}_{uvk}, \qquad (3)$$

and the product $t_l\, t_m\, t_r$ of the three partial transpositions is the matrix transposition.

All the generators are involutions. All products $t_l I$, $t_m I$ and $t_r I$ are of infinite order when acting on a generic matrix, as are $\tau_l I$ and $\tau_r I$. On the contrary τ_m and I commute and $(\tau_m I)^2 = 1$.

It is straightforward to check that:

$$t_l\, I\, t_l \;=\; \tau_l\, I\, \tau_l \qquad \text{and} \qquad t_r\, I\, t_r \;=\; \tau_r\, I\, \tau_r \qquad (4)$$

so that essentially Γ_{triang} appears as a subgroup of Γ_{cubic}, up to finite factors.

It is important to keep in mind what the "size" of the groups Γ are. All three Γ_{square}, Γ_{triang}, and Γ_{cubic} are infinite, but Γ_{square} has one infinite order generator, Γ_{triang} has two, and Γ_{cubic} has three. The last two groups are thus hyperbolic groups [7], and studying the triangular lattice can be a good test-case for the more involved tridimensional cubic lattice.

The groups Γ are the building pieces of the group of automorphisms of the Yang-Baxter equations and their higher dimensional generalizations, and solve the so-called "baxterization problem" [4, 5]. These equations form overdetermined systems of multilinear equations, of which the possible solutions are parametrized by algebraic varieties [8]. The overdetermination increases very rapidly with the dimension of the lattice. At the same time, the size of Γ also explodes. When looking at solutions of the Yang-Baxter equations and their generalizations to higher dimensional lattices, one faces a conflict between having a more and more overdetermined system and a larger and larger group of automorphisms for the set of solutions. We will show how this conflict is resolved in some 2D and known 3D solutions by a degeneration of the effective realization of the group Γ, which becomes finite.

The content of this letter is the description of a mechanism for such a degeneration, obtained by the linearization of specific elements of Γ.

We first show how the free-fermion condition on the asymmetric eight vertex model [9] falls into this scheme.

We then describe the group Γ_{triang} for the 32-vertex model on the triangular lattice. We show that the free-fermion conditions given in [10, 11] amount to linearizing the inverse I and make the realization of Γ_{triang} finite.

We finally write and discuss similar conditions for the 32-vertex model on the cubic (3D) lattice, by analyzing solutions of the tetrahedron equations [12, 13, 14].

One of the results we obtain is that free-fermion conditions should always appear as quadratic conditions, whatever the size and form of the matrix R is, and in particular whatever the dimension and geometry of the lattice are.

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There already exists an important literature about free-fermion models. We may refer to [15, 16, 17], where an exploration of the use of grassmannian variables, both for the construction and the resolution of the models, can be found. This work also motivated the interesting 3D construction of [18].

Our approach is based on a direct study of the matrix of Boltzmann weights, concentrating on the action of the symmetry group Γ, and provides another view on this class of models.

2 Some notations

At this point it is useful to introduce some notations we will use in the sequel.

We will denote the equality of two matrices R and R' up to an overall factor by $R \simeq R'$.

We always denote by t the full matrix transposition.

We will use various gauge transformations (weak graph dualities) [19], that is to say the conjugation by invertible matrices which are tensor products, also defined up to overall factors, i.e. transformations of the type

$$R \longrightarrow g_1^{-1} \otimes g_2^{-1} \otimes \ldots \otimes g_k^{-1} \cdot R \cdot \ g_1 \otimes g_2 \otimes \ldots \otimes g_k \ .$$

Define the matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5)$$

and the matrices $\sigma_{a_1 a_2 \ldots a_k}$ of size $2^k \times 2^k$ (k will be 2 for the square lattice, 3 for the triangular and cubic lattice, and so on) by:

$$\sigma_{a_1 a_2 \ldots a_k} = \sigma_{a_1} \otimes \sigma_{a_2} \otimes \ldots \sigma_{a_k} \ .$$

We denote by $\Sigma_{a_1 a_2 \ldots a_k}$ the conjugation by $\sigma_{a_1 a_2 \ldots a_k}$.

Clearly both t and I commute with all $\Sigma_{a_1 a_2 \ldots a_k}$, up to an irrelevant sign. Moreover, from the fact that $\sigma_a \ \sigma_b \ = \ \pm \ \sigma_b \ \sigma_a, \quad \forall a, b = 0, 1, 2, 3$, the gauge transformations Σ satisfy

$$\Sigma_{a_1 a_2 \ldots a_k} \ \Sigma_{b_1 b_2 \ldots b_k} \ = \ \pm \ \Sigma_{b_1 b_2 \ldots b_k} \ \Sigma_{a_1 a_2 \ldots a_k},$$

meaning that they commute up to a factor.

Particular gauge transformations of interest are

$$\pi \ = \ \Sigma_{33 \ldots 3},$$

and some transformations acting just by changes of sign of some of the entries, and denoted $\epsilon_\alpha \ (\alpha = l, \ m, \ r, \ \ldots)$.

If $k = 2$ (square lattice), then $\alpha = l$ or r, and

$$\epsilon_l = \Sigma_{30}, \qquad \epsilon_r = \Sigma_{03}.$$

If $k = 3$ (triangular and cubic lattice):

$$\epsilon_l = \Sigma_{300}, \qquad \epsilon_m = \Sigma_{030}, \qquad \epsilon_r = \Sigma_{003}.$$

3 Free-fermion asymmetric eight-vertex model

The matrix R of the asymmetric eight-vertex model [20] is of the form

$$R = \begin{pmatrix} a & 0 & 0 & d' \\ 0 & b & c' & 0 \\ 0 & c & b' & 0 \\ d & 0 & 0 & a' \end{pmatrix} \tag{6}$$

Notice that this form is the most general matrix satisfying $\pi\, R = R$.

The free-fermion condition [9] (see also [21]) is

$$aa' - dd' + bb' - cc' = 0 \tag{7}$$

A matrix of the form (6) may be brought, by similarity transformations, to a block-diagonal form

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \qquad \text{with} \qquad R_1 = \begin{pmatrix} a & d' \\ d & a' \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} b & c' \\ c & b' \end{pmatrix}.$$

If one denotes by $\delta_1 = aa' - dd'$ and by $\delta_2 = bb' - cc'$ the determinants of the two blocks then the matrix inverse I written polynomially (namely $R \to det(R) \cdot R^{-1}$) just reads

$$a \to a' \cdot \delta_2, \quad a' \to a \cdot \delta_2, \quad d \to -d \cdot \delta_2, \quad d' \to -d' \cdot \delta_2,$$
$$b \to b' \cdot \delta_1, \quad b' \to b \cdot \delta_1, \quad c \to -c \cdot \delta_1, \quad c' \to -c' \cdot \delta_1.$$

The condition (7) may be written as $p_9(R) = 0$ with the notations of [6], and is consequently left invariant by Γ_{square}. It is straightforward to see that condition (7) is $\delta_1 = -\delta_2$ and has the effect of linearizing I into

$$a \to a', \quad a' \to a, \quad d \to -d, \quad d' \to -d',$$
$$b \to -b', \quad b' \to -b, \quad c \to c, \quad c' \to c'.$$

The group Γ is then realized by permutations of the entries, mixed with changes of signs, and its orbits are thus *finite*. The commutators of partial transpositions and inversion, in the sense of group theory, i.e: $t_l I t_l^{-1} I^{-1} = (t_l\, I)^2$ and $t_r I t_r^{-1} I^{-1} = (t_r\, I)^2$ reduce to a change of sign of the non-diagonal entries of R. These commutators are typical infinite order elements of Γ, when acting on a generic matrix, and their degeneration is a key to the finiteness of the realization of Γ.

If we introduce the grading gr

$$gr(R) = \begin{pmatrix} a & 0 & 0 & d' \\ 0 & -b & -c' & 0 \\ 0 & -c & -b' & 0 \\ d & 0 & 0 & a' \end{pmatrix}$$

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which operates by changing the sign of the entries of only one of the two blocks, say R_2, then, for any R satisfying (7), the action of the inverse reduces to

$$I(R) \simeq t \, \Sigma_{12} \, gr(R) \tag{8}$$

where t is matrix transposition. In other words we have defined, on all matrices satisfying $\pi(R) = R$, a linear operator

$$l_{sq} = t \, \Sigma_{12} \, gr$$

such that the free-fermion condition (7) reads

$$I(R) \simeq l_{sq}(R) \tag{9}$$

or equivalently

$$R \cdot l_{sq}(R) \simeq \text{ unit matrix} \tag{10}$$

The linear transformation l_{sq} satisfies a number of relations:

$$l_{sq}^2 = id, \qquad l_{sq} \, t = t \, l_{sq}, \qquad l_{sq} \, t_\alpha \, l_{sq} \, t_\alpha = \epsilon_\alpha, \quad \alpha = l, \, r \tag{11}$$

Such relations ensure that the orbit of R under Γ is finite, as is readily checked, and specify the changes of signs to which $(t_l \, I)^2$ and $(t_r \, I)^2$ reduce. Notice that the definition of l_{sq} is not unique.

4 Free-fermion conditions for the 32-vertex model on the triangular lattice

We consider the free-fermion conditions for the 32-vertex model on a triangular lattice, and use the notations of [11]:

$$R = \begin{bmatrix}
f_0 & 0 & 0 & f_{23} & 0 & f_{13} & f_{12} & 0 \\
0 & f_{36} & f_{26} & 0 & f_{16} & 0 & 0 & \bar{f}_{45} \\
0 & f_{35} & f_{25} & 0 & f_{15} & 0 & 0 & \bar{f}_{46} \\
f_{56} & 0 & 0 & \bar{f}_{14} & 0 & \bar{f}_{24} & \bar{f}_{34} & 0 \\
0 & f_{34} & f_{24} & 0 & f_{14} & 0 & 0 & \bar{f}_{56} \\
f_{46} & 0 & 0 & \bar{f}_{15} & 0 & \bar{f}_{25} & \bar{f}_{35} & 0 \\
f_{45} & 0 & 0 & \bar{f}_{16} & 0 & \bar{f}_{26} & \bar{f}_{36} & 0 \\
0 & \bar{f}_{12} & \bar{f}_{13} & 0 & \bar{f}_{23} & 0 & 0 & \bar{f}_0
\end{bmatrix} \tag{12}$$

This matrix may be brought, by a permutations of lines and columns, into a block diagonal form:

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \qquad \text{with} \tag{13}$$

$$
R_1 = \begin{bmatrix} f_0 & f_{13} & f_{12} & f_{23} \\ f_{46} & \bar{f}_{25} & \bar{f}_{35} & \bar{f}_{15} \\ f_{45} & \bar{f}_{26} & \bar{f}_{36} & \bar{f}_{16} \\ f_{56} & \bar{f}_{24} & \bar{f}_{34} & \bar{f}_{14} \end{bmatrix} , \qquad
R_2 = \begin{bmatrix} f_{14} & f_{34} & f_{24} & \bar{f}_{56} \\ f_{16} & f_{36} & f_{26} & \bar{f}_{45} \\ f_{15} & f_{35} & f_{25} & \bar{f}_{46} \\ \bar{f}_{23} & \bar{f}_{12} & \bar{f}_{13} & \bar{f}_0 \end{bmatrix} \tag{14}
$$

The inverse I written polynomially is now a transformation of degree 7. If one introduces the two determinants $\Delta_1 = det(R_1)$ and $\Delta_2 = det(R_2)$, then each term in the expression of $I(R)$ is a product of a degree three minor, taken within a block, times the determinant of the other block.

Denoting $\bar{f}_{12} = f_{3456}$ and so on, the free-fermion conditions of [10, 11] are:

$$
\begin{aligned}
f_0 f_{ijkl} &= f_{ij} f_{kl} - f_{ik} f_{jl} + f_{il} f_{jk}, \qquad \forall\, i,j,k,l = 1,\ldots,6 \tag{15} \\
f_0 \bar{f}_0 &= f_{12} \bar{f}_{12} - f_{13} \bar{f}_{13} + f_{14} \bar{f}_{14} - f_{15} \bar{f}_{15} + f_{16} \bar{f}_{16} \tag{16}
\end{aligned}
$$

What is remarkable is that, not only the rational variety $\mathcal{V}$ defined by (15,16), is globally invariant by Γ_{triang}, but *again* the realization of Γ on this variety is finite. This comes from the degeneration of I into a mixture of changes of signs and permutations of the entries, as was the case in the previous section.

When relations (15, 16) are satisfied, the action of I simplifies to

$$
I(R) \simeq l_{tr}(R) \tag{17}
$$

with

$$
l_{tr} = t\, \Sigma_{121} \simeq t\, \Sigma_{111}\, \epsilon_m \tag{18}
$$

Since we have the prejudice that all free-fermion conditions should be invariant under Γ_{triang}, as (15, 16) are, one should complement (17) with

$$
\begin{aligned}
I\, \tau_l(R) &\simeq l_{tr}\, \tau_l\,(R) \tag{19} \\
I\, \tau_r(R) &\simeq l_{tr}\, \tau_r\,(R) \tag{20}
\end{aligned}
$$

We may list some useful relations:

$$
\begin{aligned}
\tau_l\, \Sigma_{abc} = \Sigma_{acb}\, \tau_l, \qquad \tau_m\, \Sigma_{abc} = \Sigma_{cba}\, \tau_m, \qquad \tau_r\, \Sigma_{abc} = \Sigma_{bac}\, \tau_r \tag{21} \\
t\, \tau_\alpha = \tau_\alpha\, t \qquad \forall\, a,b,c = 0,1,2,3,\ \forall\, \alpha = l,m,r \tag{22} \\
(\tau_\alpha\, \tau_\beta\, \tau_\gamma)^2 = id \qquad \forall \alpha,\beta,\gamma = l,m,r \tag{23} \\
\tau_l \tau_m \tau_r\, I = I\, \tau_l \tau_m \tau_r, \qquad \tau_l \tau_r \tau_l\, I = I\, \tau_l \tau_r \tau_l \tag{24}
\end{aligned}
$$

The linear transformation l_{tr} satisfies in addition:

$$
\begin{aligned}
l_{tr}^2 = id, \qquad l_{tr}\, t = t\, l_{tr} \tag{25} \\
l_{tr}\, t_l\, l_{tr} t_l = \epsilon_l, \qquad l_{tr}\, t_m\, l_{tr} t_m = id, \qquad l_{tr}\, t_r\, l_{tr} t_r = \epsilon_r \tag{26}
\end{aligned}
$$

Using relations (21) to (26), it is possible to show that the completed system (17, 19, 20) is left invariant by the action of the group Γ_{triang}.

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The system is also invariant under the gauge transformations leaving the form (12) stable. (Hint: The gauge transformations leaving the form (12) stable satisfy $^t\gamma \, \Sigma_{abc} \, \gamma \, \simeq \, \Sigma_{abc}$ when $a, b = 1, 2$).

Moreover the generic 32-vertex invertible solutions of the completed system (17, 19, 20) satisfy the free-fermion conditions (15, 16).

It is clear from (17, 19, 20) and (21) to (26) that the realization of the group Γ_{triang} is finite, when conditions (15, 16) are fulfilled.

One should also notice that any linearization condition of the type of (17) is a set of *quadratic conditions, whatever the size of the matrix is*. Indeed they mean that the matrix product of R with some linear transformed $\mathcal{L}(R)$ of R is proportional to the unit matrix, i.e:

$$R \cdot \mathcal{L}(R) \simeq \text{ unit matrix} \tag{27}$$

and this is a set of quadratic conditions.

Remark: The invertible solutions of (17) form a group for the ordinary matrix product, since $I \cdot l_{tr}$ is an automorphism of the group of invertible matrices of the form (12), i.e. $I(l_{tr}(R_1 \cdot R_2)) = I(l_{tr}(R_1)) \cdot I(l_{tr}(R_2))$. The extra conditions added when completing the system break this in such a way that the ordinary matrix product of three solutions is another solution. In other words, if R_1, R_2, R_3 $\in \mathcal{V}$, then $R_1 \cdot R_2 \cdot R_3 \in \mathcal{V}$, while $R_1 \cdot R_2 \notin \mathcal{V}$. This was actually already the case for solutions of (7), but the mechanism is more subtle here as conditions (15, 16) imply $\Delta_1 = +\Delta_2$.

5   32-vertex model on the cubic lattice

We now turn to a solution of the tetrahedron equations [14, 22, 23]. Let R be of the form

$$R = \begin{pmatrix} d & 0 & 0 & -a & 0 & -b & c & 0 \\ 0 & w & x & 0 & y & 0 & 0 & z \\ 0 & x & w & 0 & z & 0 & 0 & y \\ -a & 0 & 0 & d & 0 & c & -b & 0 \\ 0 & -y & z & 0 & w & 0 & 0 & -x \\ b & 0 & 0 & c & 0 & d & a & 0 \\ c & 0 & 0 & b & 0 & a & d & 0 \\ 0 & z & -y & 0 & -x & 0 & 0 & w \end{pmatrix} \tag{28}$$

The form of (28) is stable [1] under the group Γ_{cubic}, and it is natural to look for invariants of Γ in the space of parameters $\{a, b, c, d, x, y, z, w\}$ [24]. There exist five algebraically independent quadratic polynomials in the entries, transforming

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[1] Notice that the form (28) is not stable by the circular permutation of the three spaces $\{l, m, r\}$.

covariantly, and with the same covariance factors under all generators of Γ_{cubic}. They are:

$$ax, \quad by, \quad cz, \quad dw, \quad \text{and} \quad Q = a^2 + c^2 - d^2 - y^2 - b^2 + x^2 - w^2 + z^2 \ .$$

We thus have four algebraically independent invariants of Γ_{cubic}, say for example

$$\chi_1 = \frac{ax}{dw}, \quad \chi_2 = \frac{by}{dw}, \quad \chi_3 = \frac{cz}{dw}, \quad \text{and} \quad \chi_0 = \frac{Q}{dw}.$$

A complete analysis shows that there is no other algebraically independent invariant of Γ_{cubic}. A numerical and graphical study [25], shows how "big" the realization of Γ is for generic values of the above invariants.

These invariants are completely specified in the solution [14], for which

$$\chi_1 \ = \ \chi_2 \ = \ \chi_3 \ = \ 1 \tag{29}$$

$$\chi_0 = 0 \tag{30}$$

Out of the four invariants, χ_0 plays a special role. *If $\chi_0 = 0$, then the action of I linearizes quite in the same way as in the previous cases.* Notice that, strictly speaking, condition (30) is not so much an assignment of value to the invariant χ_0 but rather a vanishing condition for the covariant quantity Q. Recall that assigning a definite value to an invariant object is meaningful whatever this value is. On the contrary covariant objects cannot be assigned a value unless this value is zero.

When $Q = 0$, one gets

$$I(R) \ \simeq \ l_c(R) \tag{31}$$

The linear transformation l_c may be written

$$l_c = \ t \ \Sigma_{030} \ gr,$$

where t is transposition and gr is a grading changing the sign of the entries of R belonging to the same block, say $\{x, y, z, w\}$. Notice that the definition of l_c is not unique, due to the very specific form of (28). Notice also that $Q = 0$ is one of two quadratic conditions ensuring the equality of the determinants of the two blocks of R (see (13)). The other one is not stable under Γ.

The linear transformation l_c satisfies

$$l_c^2 = id, \qquad l_c \ t = t \ l_c, \tag{32}$$

$$l_c \ t_\alpha \ l_c t_\alpha = \epsilon_\alpha, \qquad \forall \alpha = l, \ m, \ r \tag{33}$$

Any matrix of the form (28) with $Q = 0$ obeys

$$I(R) \ \simeq \ l_c(R) \tag{34}$$

$$I \ t_\alpha \ (R) \ \simeq \ l_c \ t_\alpha \ (R) \qquad \forall \alpha = l, m, r \tag{35}$$

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Using (32,33), it is straightforward to show that the complete system (34,35) is invariant under Γ_{cubic} and that the orbit of R is finite.[2]

The study of the additional conditions (29) would take us beyond the scope of this letter, but we may make a few remarks.

The first remark is that since among conditions (29,30), only (30) has to do with the finiteness of the realization of Γ, (29) may have nothing to do with free-fermion conditions. They are *additional constraints* making the resolution of the tetrahedron equations possible, and this may be understood as follows.

The tetrahedron equations are in essence a compatibility condition for the existence of non-trivial solutions of the "propagation properties" [27] (alias "Zamolodchikov algebra" [28], alias "vacuum curves" [29, 22], alias "pre-Bethe Ansatz" equations [6, 30]):

$$R \begin{pmatrix} 1 \\ p \end{pmatrix} \otimes \begin{pmatrix} 1 \\ q \end{pmatrix} \otimes \begin{pmatrix} 1 \\ r \end{pmatrix} \simeq \begin{pmatrix} 1 \\ p' \end{pmatrix} \otimes \begin{pmatrix} 1 \\ q' \end{pmatrix} \otimes \begin{pmatrix} 1 \\ r' \end{pmatrix} \tag{36}$$

What conditions (29,30) ensure is the existence, for fixed R, of a *one-parameter family of solutions* of (36). In the case we consider here, the family happens to be parametrized by a *curve of genus larger than one*.

By eliminating $\{q, q', r, r'\}$ (resp. $\{p, p', r, r'\}$ or $\{p, p', q, q'\}$) from (36), one gets conditions relating p, p', (resp. q, q' and r, r'). Such relations are generically of degree 8 (biquartics). One effect of (29,30) is that they all reduce to *asymmetric* biquadratic relations, defining three genus one curves of the form

$$x^2 y^2 - 1 + (y^2 - x^2)\, \kappa_{xy} = 0 \tag{37}$$

with $\quad \kappa_{pp'} = \dfrac{bc\,(d^2 - a^2)}{ad\,(b^2 - c^2)}, \quad \kappa_{qq'} = -\dfrac{ac\,(b^2 + d^2)}{bd\,(a^2 + c^2)}, \quad \kappa_{rr'} = \dfrac{ab\,(c^2 - d^2)}{cd\,(a^2 - b^2)}.$

These three elliptic curves have different (algebraically independent) moduli. Their asymmetric character may be an obstacle to the use of (36) in the construction of the Bethe Ansatz states [3], since the composition of relations of type (37) reproduces the same type of relations, but alters the value of κ by:

$$\kappa \longrightarrow \frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right) \tag{38}$$

Exceptional values of κ $(\pm 1, \infty)$, yielding a rationalization of (37), are fixed points of (38). For these exceptional values, in particular $\kappa = \infty$, obtained with $d = 0$, the construction of a 3D Bethe Ansatz may be envisaged.

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[2]This is also the case for the bidiagonal solution of the "constant" tetrahedron equations of [26].

[3]R.J. Baxter, private communication.

6   Conclusion

We have shown, through specific examples, how free–fermion conditions turn into degeneration conditions of our groups Γ: the generically non-linear (rational) infinite realization of Γ becomes a linear finite group.

We believe this is a characteristic feature of 2D free-fermion models.

We have shown that the known vertex solution of the tetrahedron equations does have such a feature. An appealing issue is to decide whether or not such a statement can be made about other 3D and higher dimensional models. Of course the full answer will come from linking directly the phenomenon we describe with explicit calculus using grassmannian variables. The particularly simple form of the conditions (combinations of products of entries with plus and minus signs), and the linearization process of the inverse should stem from elementary properties of exponentials of quadratic forms in anticommuting variables.

Producing new solutions of the tetrahedron equations is another challenging problem. What could be done is to look for forms of the matrices R enjoying the linearization property we have described. This is a rather simple way to produce "reasonable" Ansätze for R.

The next step would then be to study the so-called propagation properties (see above) rather than confronting directly the tetrahedron equations themselves.

Indeed these simpler equations, because they govern the construction of Bethe Ansatz states –a basic in the field–, underpin 2D, 3D, and higher dimensional integrability.

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